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ANALYSIS OF AXISYMMETRICAL VIBRATION OF A PARTIALLY LIQUID-FILLED ELASTIC SPHERE BY THE METHOD OF GREEN'S FUNCTION

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16. ABSTRACT

The longitudinal dynamic behavior of launch vehicles is largely determined by their huge liquid propellant masses which are spring supported by the elastic tank walls. Throughout much of the powered flight time, these masses constitute a high percentage of the entire vehicle mass and, therefore, may dominate the fundamental modes of the vehicle.

In this report, a spherical container is considered. The analysis is based on a Galerkin approach, in the course of which a second-order differential equation must be solved. The solution has been obtained by the method of Green's function. This method is favorable because it displays the manner in which the analysis can be extended to partially liquid-filled general shells of revolution.

The computer programs currently available for partially liquid-filled propellant tanks are based on the finite element methods and result in analytical models having as many as several hundred degrees of freedom. The method applied in this report results in a model having less than 10 degrees of freedom as can be shown by numerical evaluation. Therefore, it will be possible to analyze propellant tanks using much less computer time with comparable accuracy.

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DEFINITION OF SYMBOLS

Symbol | Definition

$A_k(\psi)$ quantity defined by equation (A-6)

$A_k(\phi_1,\phi)$ integral defined by equation (41)

a   radius of sphere

a_k Dimensionless coefficient defined by equation (7)

$\begin{bmatrix} a_{j1} \\ a_{j2} \\ \vdots \\ a_{jn} \end{bmatrix}$ jth eigenvector

$C_k(\phi)$ quantity defined by equation (A-9)

$C_k(\phi_1,\phi)$ integral defined by equation (42)

$\overline{D}$ quantity defined by equation (20c)

E   Young's modulus of elasticity

f   function symbol

f_j functions defined by equations (A-3) and (A-4)

F_k function defined by equation (A-5)

$G(\phi,\psi)$ Green's function given by equations (27) and (28)

h   Coordinate of the free liquid surface with respect to the center O of the sphere

$J_0(r)$ Bessel function of the order zero

N_θ, N_ϕ membrane forces of the spherical shell

O   center of the sphere; see Figure 1

DEFINITION OF SYMBOLS (Continued)

Symbol	Definition
P	Symbol of Legendre's polynomial appearing in equations (37)
P_j	Legendre's polynomial of the first kind of degree j
p	liquid pressure
Q_1	Legendre's polynomial of the second kind of order one
t	time
v	wall displacement; see Figure 1
$v^{(j)}$	wall displacement belonging to the jth mode of vibration
v_k	coefficient defined by equations (11a)
W	wetted wall of the sphere
w	wall displacement; see Figure 1
$w^{(j)}$	wall displacement belonging to the jth mode of vibration
w_k	coefficient defined by equations (11b)
x	real variable
γ	mass density of the liquid
δ	thickness of the shell wall
$\epsilon_\theta, \epsilon_\phi$	strains
θ	spherical coordinate
κ_{jk}	matrix coefficient defined by equation (13)
λ	parameter defined by equation (9)
λ_j	eigenvalue defined by equation (16)

DEFINITION OF SYMBOLS (Concluded)

Symbol	Definition
μ_{jk}	matrix coefficient defined by equations (14)
ν	Poisson's constant
$\xi(\phi)$	function defined by equations (24) and (25)
$\xi_k(\phi)$	function defined by equation (30)
ρ	spherical coordinate
τ	region of the sphere occupied by the liquid
$\Phi(\rho,\phi,t)$	velocity potential
$\Phi_k(\rho,\phi)$	potential function defined by equations (37)
ϕ,ψ,ϕ_1	spherical coordinates
ϕ_s	spherical coordinate of the liquid surface
ω	parameter defined by equation (7)
ω_i	ith circular eigenfrequency defined by equation (16)

ANALYSIS OF AXISYMMETRICAL VIBRATION OF A PARTIALLY LIQUID-FILLED ELASTIC SPHERE BY THE METHOD OF GREEN'S FUNCTION

INTRODUCTION

The longitudinal dynamic behavior of launch vehicles is largely determined by their huge liquid propellant masses which are spring supported by the elastic tank walls. Throughout much of the powered flight time, these masses constitute a high percentage of the entire vehicle mass and, therefore, may dominate the fundamental modes of the vehicle. Consequently, the vibration analysis of partially liquid-filled tanks is of importance.

In this report, a spherical container [1,2] is considered. The analysis, however, can easily be extended to cylindrical containers having spherical bulkheads [2]. As in References 1 and 2, the analysis is based on a Galerkin approach, in the course of which a second-order differential equation must be solved. Differing from the analyses of References 1 and 2, the solution is obtained by the method of Green's function. This method is favorable from two points of view. On the one hand, its application simplifies the adaption to the boundary conditions. On the other hand, this method displays the manner in which the analysis can be extended to partially liquid-filled membrane shells of revolution. This will be shown, however, in a subsequent report.

A few remarks concerning the method of Galerkin seem to be appropriate. Like the Ritz method, this method nowadays is considered to be a convenient tool for the solution of eigenvalue problems. As is well known, the Galerkin approach leads to a matrix eigenvalue problem in which the order equals the number of used coordinate functions. Increasing the number of coordinate functions step by step results in sequences of approximate eigenvalues. Now, the point is that the problem at hand belongs to a wider class of problems where the sequences of approximate eigenvalues converge to the exact eigenvalues [3]. Obviously, the quality of the convergence depends upon the coordinate functions selected. For the case at hand, however, the convergence is excellent, as can be demonstrated by numerical evaluation.

The computer programs currently available for partially liquid-filled propellant tanks are based on the finite element methods and result in analytical models having as many as several hundred degrees of freedom. The method applied in this report results in a model having less than 10 degrees of freedom; therefore, it will be possible to analyze propellant tanks using much less computer time and with comparable accuracy.

Based on this analytical model, a computer program and numerical results will be reported later.

VIBRATING SYSTEM AND BASIC ASSUMPTIONS

Figure 1 shows a meridian plane of the partially liquid-filled sphere. The liquid-filled region and the wetted wall are denoted by τ and W, respectively. The displacements of the wall are v in the direction of the meridian tangent and w in direction of the outer normal to the wall.
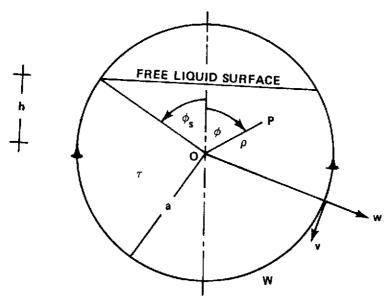

Figure 1. Vibrating system, wall displacements.

The vibrating system is referred to a spherical coordinate system $P(\rho,\phi,\theta)$ having its origin O in the center of the sphere (Fig. 1). θ is the longitude. The sphere is supported along the equator. Accordingly, the support condition is

$$\phi = \frac{\pi}{2} \quad , \quad v = 0 \quad .$$

The distance of the free surface from the center O is

$$h = a \cos \phi_s \quad . \tag{1}$$

In the chosen coordinates, the equation of the free liquid surface is

$$h = \rho \cos \phi \quad . \tag{2}$$

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The assumptions upon which the following analysis is based are as usual. Because the mass of the wall is assumed to be small compared with the liquid mass inside the sphere, it will be neglected. Also neglected will be the axial bending as to the small effect on the fundamental frequencies. In summary, the container wall is interpreted to behave like a massless membrane shell.

To describe the liquid motion which is assumed to be small, the existence of a velocity potential Φ is assumed. Thus,

$$\nabla^2 \Phi(\rho,\phi,t) = 0 \tag{3}$$

within the liquid-filled region τ of the container. Then the pressure in τ follows as

$$p = -\gamma \frac{\partial \Phi}{\partial t} \qquad . \tag{4}$$

The boundary condition at the wetted wall W is

$$\left. \frac{\partial \Phi}{\partial \rho} \right|_{\rho = a} = \frac{\partial w}{\partial t} \qquad . \tag{5}$$

At the free surface, the pressure caused by the small surface elevation is neglected, i.e.,

$$\Phi(\rho,\phi,t) = 0 \tag{6}$$

for ρ and ϕ satisfying equation (2).

EIGENVALUE PROBLEM

The potential Φ will be determined by a Galerkin approach [3] proposed in Reference 2 and also employed in Reference 1. Using a set of linear independent functions

$$\Phi_j(\rho,\phi) \qquad ; \qquad j = 1, 2, 3, \dots$$

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which are harmonic within the region τ of the sphere and satisfy the free surface condition, equation (6), Φ is to be approximated by

$$\Phi(\rho,\phi,t) = a^2 \omega \cos\omega t \sum_{k=1}^{n} a_k \Phi_k(\rho,\phi) \qquad (7)$$

where the coefficients a_k and the functions Φ_k are assumed to be dimensionless.

The coefficients a_k will be determined by imposing a set of orthogonality conditions on the error function which results from the substitution of the assumed solution into the boundary condition, equation (5), i.e.,

$$\int_W \left(\frac{\partial\Phi}{\partial\rho} - \frac{\partial w}{\partial t} \right) \Phi_j dW = 0 \quad ; \quad j = 1, 2, 3, ... n \qquad . \qquad (8)$$

For convergence it is necessary to select the functions Φ_j such as to form the first n functions of a system which is complete on W.

Now w must be considered. From equations (4) and (7) it is realized that

$$p = \gamma a^2 \omega^2 \sin\omega t \sum_{k=1}^{n} a_k \Phi_k$$

or if

$$\lambda = \frac{a^3 \gamma \omega^2}{E\delta} \qquad (9)$$

is substituted,

$$p = \frac{E\delta}{a} \lambda \sum_{k=1}^{n} a_k \Phi_k \qquad (10)$$

The time factor of equation (10) and also of the following equations has been omitted.

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The displacements v and w are caused by the pressure acting on the wall. Therefore, the following expansions can be concluded

$$v = a \lambda \sum_{k=1}^{n} a_k v_k \qquad (11a)$$

and

$$w = a \lambda \sum_{k=1}^{n} a_k w_k \qquad (11b)$$

where the quantities v_k and w_k depend on Φ_k.

Substitution of equation (7) and (11b) into equation (8) results in

$$\sum_{k=1}^{n} a_k \left(a \int_W \Phi_j \frac{\partial \Phi_k}{\partial \rho} dW - \lambda \int_W \Phi_j w_k dW \right) = 0 \quad ; \quad j = 1, 2, 3, \ldots \quad . \quad (12)$$

If the entire equation (12) is divided by $2a^2\pi$ and the notations

$$\frac{1}{2a\pi} \int_W \Phi_j \frac{\partial \Phi_k}{\partial \rho} dW = \kappa_{jk} \qquad (13)$$

and

$$\frac{1}{2a^2\pi} \int_W \Phi_j w_k dW = \mu_{jk} \qquad (14)$$

are introduced, the equations (12) can be written as

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$$\left(\begin{bmatrix} \kappa_{11} & \kappa_{12} & \cdots & \kappa_{1n} \\ \kappa_{21} & \kappa_{22} & \cdots & \kappa_{2n} \\ \vdots & \vdots & & \vdots \\ \kappa_{n1} & \kappa_{n2} & \cdots & \kappa_{nn} \end{bmatrix} - \lambda \begin{bmatrix} \mu_{11} & \mu_{12} & \cdots & \mu_{1n} \\ \mu_{21} & \mu_{22} & \cdots & \mu_{2n} \\ \vdots & \vdots & & \vdots \\ \mu_{n1} & \mu_{n2} & \cdots & \mu_{nn} \end{bmatrix} \right) \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = 0 \quad . \quad (15)$$

In this way the problem at hand is reduced to a matrix eigenvalue problem, the solutions of which are denoted by

$$\lambda_i = \frac{a^3 \gamma \omega_i^2}{E\delta} \quad ; \quad \begin{bmatrix} a_{i1} \\ a_{i2} \\ \cdot \\ \cdot \\ a_{in} \end{bmatrix} \quad ; \quad i = 1, 2, \ldots \quad . \quad (16)$$

Using the spherical coordinate system (ρ, θ, ϕ) mentioned in the beginning, the surface element of the wall is

$$dW = a^2 \sin\phi \, d\phi \, d\theta \quad .$$

Thus, equations (13) and (14) can be written

$$\kappa_{jk} = a \int_{\phi_S}^{\pi} \Phi_j(a, \phi) \left. \frac{\partial \Phi_k}{\partial \rho} \right|_{\rho=a} \sin\phi \, d\phi \quad (17)$$

$$\mu_{jk} = \int_{\phi_S}^{\pi} \Phi_j(a, \phi) \, w_k \sin\phi \, d\phi \quad (18)$$

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To solve the eigenvalue problem (15), the matrix coefficients (17) and (18) must be determined. The coefficient (17) depends on the coordinate functions only. Once the coordinate functions have been selected, κ_{jk} can immediately be determined. In the case of μ_{jk}, the situation is different because it depends on the displacement w_k. The wall displacements depend on the pressure exerted by the liquid at the wetted container wall. As mentioned previously, the wall is assumed to behave like a membrane shell. Hence, to obtain the equations that link the displacements to the pressure, the basic equations of the membrane shell theory must be presented [4].

MEMBRANE SHELL EQUATIONS

The equilibrium conditions are

$$N_\phi + N_\theta = ap \tag{19a}$$

and

$$\frac{d}{d\phi} (N_\phi \sin\phi) - N_\theta \cos\phi = 0 \quad . \tag{19b}$$

Hooke's law is given by

$$N_\theta = \overline{D}(\nu\epsilon_\phi + \epsilon_\theta) \tag{20a}$$

and

$$N_\phi = \overline{D}(\epsilon_\phi + \nu\epsilon_\theta) \tag{20b}$$

where

$$\overline{D} = \frac{E\delta}{1 - \nu^2} \quad . \tag{20c}$$

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Finally the strain-displacement relations are

$$v \cot\phi + w = a \, \epsilon_\theta \tag{21a}$$

and

$$\frac{dv}{d\phi} + w = a \, \epsilon_\phi \tag{21b}$$

The positive directions of v and w can be seen in Figure 1.

By substituting equations (20a) through (21b) into equations (19) [1,2], one obtains

$$v = -(1 + \nu) \frac{a^2}{E\delta} \frac{d\xi}{d\phi} \tag{22}$$

and

$$w = \frac{a^2}{E\delta} [p - (1 + \nu) \xi] \tag{23}$$

where

$$\frac{d}{d\phi} \left(\sin\phi \, \frac{d\xi}{d\phi} \right) + 2 \sin\phi \, \xi = \sin\phi \, p \tag{24}$$

Equation (24) is a differential equation of second order. For the pressure p to determine the displacements v and w, this equation must be solved under consideration of proper boundary conditions. From equation (22), these conditions can be concluded as

$$\frac{d\xi}{d\phi} = 0 \quad \text{at} \quad \phi = 0, \frac{\pi}{2}, \pi \tag{25}$$

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The conditions at $\phi = 0, \pi$ follow from the axisymmetry of the problem while the conditions at $\phi = \pi/2$ represents the support conditions as noted in the beginning.

GREEN'S FUNCTION

Equation (24) will be solved by the use of Green's function $G(\phi, \psi)$. According to Reference 5, the left side of equation (24) is a self-adjoint differential expression, hence $G(\phi, \psi)$ is a symmetric function. To construct Green's function [5], one has to start from the solutions of the homogeneous equation (24). Because this equation is Legendre's differential equation of the degree 1, two linear independent solutions [6] are given by

$$\cos\phi \text{ and } Q_1(\cos\phi) = -1 - \cos\phi \ln \tan \frac{\phi}{2} \tag{26}$$

Green's function is linear, composed of these solutions, and must satisfy the boundary conditions of equation (25) at $\phi = 0$, $\pi/2$, and π. Hence, it follows that two independent Green functions exist for the upper $(0 \leq \phi \leq \pi/2)$ and for the lower hemisphere $(\pi/2 \leq \phi \leq \pi)$, respectively. No influence can be transferred from the upper to the lower hemisphere and vice versa. Furthermore [5], $G(\phi, \psi)$ is continuous at $\phi = \psi$, whereas its first derivative has a jump discontinuity at that place in the amount of $1/\sin\phi$. Accordingly, Green's function is

$$\phi \leq \psi \qquad G(\phi, \psi) = -Q_1(\cos\psi) \cos\phi \tag{27a}$$

$$\phi \geq \psi \qquad G(\phi, \psi) = -Q_1(\cos\phi) \cos\psi \tag{27b}$$

for the upper hemisphere $(0 \leq \phi \leq \pi/2)$ and

$$\phi \leq \psi \qquad G(\phi, \psi) = Q_1(\cos\phi) \cos\psi \tag{28a}$$

$$\phi \geq \psi \qquad G(\phi, \psi) = Q_1(\cos\psi) \cos\phi \tag{28b}$$

for the lower hemisphere $(\pi/2 \leq \phi \leq \pi)$.

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Thus, from equation (24) it follows for the wetted wall

$$\xi(\phi) = \int_{\phi_1}^{\phi_2} G(\phi,\psi)\, p\, \sin\psi\, d\psi \tag{29}$$

where $G(\phi,\psi)$ is given by equations (27) and (28). In the case of $\phi_S < \pi/2$, the limits of the integral are

$$\langle \phi_1, \phi_2 \rangle = \begin{cases} \langle \phi_S, \pi/2 \rangle \\[2ex] \langle \pi/2, \pi \rangle \end{cases} \tag{29a}$$

when

$$\phi < \pi/2 \qquad \text{or} \qquad \phi \geq \pi/2$$

and, in case of $\phi_S \geq \pi/2$,

$$\langle \phi_1, \phi_2 \rangle = \langle \phi_S, \pi \rangle \qquad . \tag{29b}$$

Substitution of equation (10) into equation (29) yields

$$\xi(\phi) = \frac{E\delta}{a}\, \lambda \sum_{k=1}^{n} a_k \int_{\phi_1}^{\phi_2} G(\phi,\psi)\, \Phi_k(a,\psi)\, \sin\psi\, d\psi \qquad .$$

Now, if it is assumed that

$$\xi(\phi) = \frac{E\delta}{a}\, \lambda \sum_{k=1}^{n} a_k\, \xi_k(\phi) \qquad , \tag{30}$$

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then

$$\xi_k(\phi) = \int_{\phi_1}^{\phi_2} G(\phi,\psi) \; \Phi_k(a,\psi) \; \sin\psi \; d\psi \tag{31}$$

where the integral has the limits of equations (29a) or (29b) when ϕ_s and ϕ lie in the first, in the first and in the second, or both in the second quadrant. Green's function is given by equations (27) and (28).

Combining equations (10), (11b), (23) and (30), one obtains for the wetted wall

$$w_k = \Phi_k(a,\phi) - (1 + \nu) \; \xi_k(\phi) \qquad . \tag{32}$$

Finally from equations (18) and (32), it is concluded that

$$\mu_{jk} = \int_{\phi_s}^{\pi} \Phi_j(a,\phi) \; \Phi_k(a,\phi) \; \sin\phi \; d\phi - (1 + \nu) \int_{\phi_s}^{\pi} \Phi_j(a,\phi) \; \xi_k(\phi) \; \sin\phi \; d\phi \tag{33}$$

where ξ_k is given by the integral (31).

COORDINATE FUNCTIONS

As mentioned previously, the solution of the problem at hand is based upon a set of potential functions Φ_n which satisfy the free surface conditions (6) and form a complete system on the wetted wall W . It is appropriate to start with the following set [6] of potential functions

$$\rho^m \; P_m(\cos\phi) \qquad ; \qquad m = 1, 2, 3, \dots \tag{34}$$

where $P_m(x)$ is Legendre's polynomial or Legendre's function of the first kind and of the degree m . For m = 0,1,2,... Legendre's polynomials [7] are

$$P_0 = 1$$

$$P_1 = x$$

$$P_2 = \frac{1}{2}(3x^2 - 1)$$

$$P_3 = \frac{1}{2}(5x^3 - 3x)$$

and so forth.

The coordinate functions Φ_k will be linear composed of the functions (34) so as to fulfil the free surface condition (6). For that purpose it is convenient to present the generating function [7] of the system (34)

$$e^{\rho\cos\phi}\, J_0(\rho,\sin\phi) = \sum_{m=0}^{\infty} P_m(\cos\phi)\frac{\rho^m}{m!} \tag{35}$$

where J_0 is the Bessel function of the first kind and order zero. Multiplying equation (35) by

$$e^{-h} = \sum_{m=0}^{\infty} \frac{(-h)^m}{m!}$$

results in

$$e^{\rho\cos\phi-h}\, J_0(\rho\,\sin\phi) = \sum_{m=0}^{\infty}\left[\sum_{j=0}^{m}\binom{m}{j}P_j(\cos\phi)\left(\frac{-h}{\rho}\right)^{m-j}\right]\frac{\rho^m}{m!} \tag{36}$$

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The terms of the infinite series (36) represent linear combinations of the potential functions (34). From these terms, the proper coordinate functions can be selected as is explained below. According to equation (2), equation (36) reduces to the power series of J_0 for all pairs (ρ, ϕ) of the free liquid surface. Because J_0 contains even order terms, only the odd order terms must be zero at the points (ρ, ϕ) of the liquid surface. Consequently, the odd order terms of the series (36), apart from a constant factor, can be considered the desired coordinate functions, i.e.,

$$\Phi_n(\rho,\phi) \;=\; \left(\frac{\rho}{a}\right)^{2n-1} \sum_{j=0}^{2n-1} \binom{2n-1}{j} P_j(\cos\phi) \left(-\frac{h}{\rho}\right)^{2n-1-j} \qquad ,$$

or by symbolic operations

$$\Phi_n(\rho,\phi) \;=\; \left(\frac{\rho}{a}\right)^{2n-1} \left(P - \frac{h}{\rho}\right)^{2n-1} \qquad n = 1, 2, 3, \ldots \tag{37a}$$

where

$$P^j \;=\; P_j(\cos\phi) \qquad . \tag{37b}$$

According to equation (1),

$$\Phi_n(a,\phi) \;=\; (P - \cos\phi_s)^{2n-1} \qquad . \qquad n = 1, 2, 3, \ldots \tag{37c}$$

For later use, the derivatives of the coordinate functions at the wall in the direction of the outer normal will be determined. From equation (37) it is realized that

$$a\frac{\partial\Phi_n}{\partial\rho}\bigg|_{\rho=a} \;=\; (2n-1)\, P_1 \left(P - \cos\phi_s\right)^{2n-2} \qquad ; \qquad n = 1, 2, \ldots \tag{38}$$

where

$$P_1\, P^j \;=\; P_{j+1}(\cos\phi) \qquad .$$

MATRIX ELEMENTS

The matrix coefficients κ_{jk} and μ_{jk} are given by equations (17) and (33). The coefficient κ_{jk} depends on the coordinate functions (37) and their normal derivatives (38), only and, therefore, can be easily evaluated. In the case of μ_{jk}, the situation is different because Green's function is involved, as is demonstrated by equations (31) and (33). First, Green's function as given by equations (26) through (28) must be substituted into equation (31). For the wetted wall, the result is that

$$\xi_k(\phi) = -A_k(\phi_s,\phi)\, Q_1(\cos\phi) - C_k\left(\phi,\frac{\pi}{2}\right)\cos\phi \qquad \text{if} \qquad \phi_s \leqslant \phi \leqslant \frac{\pi}{2} \tag{39a}$$

and

$$\xi_k(\phi) = C_k\left(\frac{\pi}{2},\phi\right)\cos\phi + A_k(\phi,\pi)\, Q_1(\cos\phi) \qquad \text{if} \qquad \frac{\pi}{2} \leqslant \phi \leqslant \pi \quad , \tag{39b}$$

whereas

$$\xi_k = C_k(\phi_s,\phi)\cos\phi + A_k(\phi,\pi)\, Q_1(\cos\phi) \qquad \text{if} \qquad \frac{\pi}{2} < \phi_s \leqslant \phi \leqslant \pi \quad . \tag{40}$$

Thereby

$$A_k(\phi_1,\phi) = \int_{\phi_1}^{\phi} \Phi_k(a,\psi)\cos\psi\,\sin\psi\,d\psi \tag{41}$$

and

$$C_k(\phi_1,\phi) = \int_{\phi_1}^{\phi} \Phi_k(a,\psi)\, Q_1(\cos\psi)\sin\psi\,d\psi \quad . \tag{42}$$

The evaluation of the integrals (41) and (42) is shown in the Appendix.

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Now, from equations (33), (41), and (42), it follows in the case of $\phi_S \leqslant \pi/2$ that

$$\mu_{jk} = \int_{\phi_S}^{\pi} \Phi_j(a,\phi)\, \Phi_k(a,\phi)\, \sin\phi\; d\phi$$

$$-(1+\nu)\left[-\int_{\phi_S}^{\pi/2} \Phi_j(a,\phi)\, A_k(\phi_S,\phi)\, Q_1(\cos\phi)\, \sin\phi\; d\phi \right.$$

$$-\int_{\phi_S}^{\pi/2} \Phi_j(a,\phi)\, C_k(\phi,\pi/2)\, \cos\phi\, \sin\phi\; d\phi$$

$$+\int_{\pi/2}^{\pi} \Phi_j(a,\phi)\, C_k(\pi/2,\phi)\, \cos\phi\, \sin\phi\; d\phi$$

$$\left. +\int_{\pi/2}^{\pi} \Phi_j(a,\phi)\, A_k(\phi,\pi)\, Q_1(\cos\phi)\, \sin\phi\; d\phi \right] \quad,$$

or, if partial integration is applied to the third and fourth of the above integrals,

$$\mu_{jk} = \int_{\phi_S}^{\pi} \Phi_j(a,\phi)\, \Phi_k(a,\phi)\, \sin\phi\; d\phi$$

$$+(1+\nu)\left\{ \int_{\phi_S}^{\pi/2} \left[\Phi_j(a,\phi)\, A_k(\phi_S,\phi) + A_j(\phi_S,\phi)\, \Phi_k(a,\phi) \right] Q_1(\cos\phi)\, \sin\phi\; d\phi \right.$$

$$\left. -\int_{\pi/2}^{\pi} \left[\Phi_j(a,\phi)\, A_k(\phi,\pi) + A_j(\phi,\pi)\, \Phi_k(a,\phi) \right] Q_1(\cos\phi)\, \sin\phi\; d\phi \right\} \quad.$$

$$(43)$$

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In the case of $\phi_s \geqslant \pi/2$, one obtains similarly

$$\mu_{jk} = \int_{\phi_s}^{\pi} \Phi_j(a,\phi) \, \Phi_k(a,\phi) \, \sin\phi \; d\phi$$

$$- (1 + \nu) \int_{\phi_s}^{\pi} \left[\Phi_j(a,\phi) \, A_k(\phi,\pi) + A_j(\phi,\pi) \, \Phi_k(a,\phi) \right] Q_1(\cos\phi) \sin\phi \; d\phi \quad . \quad (44)$$

Thereby the coordinate functions Φ_j are defined by equations (37), whereas the functions A_k are given by equations (A-3) through (A-7) and (A-10) of the Appendix. $Q_1(\cos\phi)$, given by equation (26), is Legendre's function of the second kind and of the degree one.

MODES OF VIBRATION

The characteristic pattern assumed by the system vibrating in its jth natural mode as defined by equation (16) are determined by its displacements $v^{(j)}$ and $w^{(j)}$. According to equations (11) and (16)

$$v^{(j)} = a \, \lambda_j \sum_{k=1}^{n} a_{jk} \, v_k \tag{45}$$

$$w^{(j)} = a \, \lambda_j \sum_{k=1}^{n} a_{jk} \, w_k \quad . \tag{46}$$

The quantity w_k for the wetted wall is given by equations (32), (37), (39), and (40). For the dry wall ($\phi \leqslant \phi_s$), equation (32) results in

$$w_k = - (1 + \nu) \, \xi_k \tag{47}$$

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where ξ_k follows from equations (27), (28), (31), (41), and (42) as

$$\xi_k(\phi) = -C_k\left(\phi_s, \frac{\pi}{2}\right)\cos\phi \qquad (48)$$

and

$$\xi_k(\phi) = A_k(\phi_s, \pi)\, Q_1(\cos\phi) \qquad (49)$$

for the upper and lower hemispheres, respectively.

From equations (11a), (22), and (30), it can be concluded that

$$v_k(\phi) = -(1 + \nu)\frac{d\xi_k}{d\phi} \qquad (50)$$

Table 1 shows $w_k(\phi)$, $v_k(\phi)$, $\xi_k(\phi)$ and $d\xi_k(\phi)/d\phi$ in cases $\phi_s \lessgtr \pi/2$ at the dry and wetted parts of the wall according to equations (32), (39), (40), and (47) through (50). The functions $A_k(\phi_1,\phi_2)$ and $C_k(\phi_1,\phi_2)$ are defined by equations (A-3) through (A-11).

CONCLUSION

Although some of the characteristics of the analysis at hand have already been discussed in the introduction, a few supplementary remarks should be added to emphasize the particular steps of the analysis. The problem to be solved is defined by the differential equation (3) and the boundary conditions (5) and (6). After selection of proper coordinate functions, the Galerkin method can be applied. In that way, the problem reduces to the matrix eigenvalue problem (15) having the matrix elements κ_{jk} and μ_{jk} as given by equations (17), (43), and (44), respectively. Once these elements are determined, the solutions (16) of the eigenvalue problem (15) can easily be obtained. Consequently, the evaluation of the integrals (17), (43), and (44) must be considered the essential steps of the analysis.

The numerical evaluation of the matrix elements κ_{jk} and μ_{jk}, however, implies no basic difficulties. The element κ_{jk} of equation (17) depends on the coordinate

functions only and once these functions are selected the integration (17) can easily be performed. To obtain the element μ_{jk}, the integrations according to equations (43) or (44), respectively, must be performed. Because all the functions involved can be represented simply by Legendre's and trigonometric functions as is concluded from equations (26), (37), (A-3) through (A-7), and (A-10), no basic integration problems exist.

Having solved the eigenvalue problem (15), one arrives at the determination of the modes of vibration. From the solutions (16) and the functions gathered in Table 1, the modes of vibration follow according to equations (45) and (46).

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TABLE 1. $w_k(\phi)$, $v_k(\phi)$, $\xi_k(\phi)$ AND $d\xi_k(\phi)/d\phi$ FOR CASES $\phi_s \lessgtr \frac{\pi}{2}$

		$w_k(\phi)$	$v_k(\phi)$	$\xi_k(\phi)$	$d\xi_k(\phi)/d\phi$
$\phi_s < \frac{\pi}{2}$	Upper Hemisphere Dry: $0 \leqslant \phi \leqslant \phi_s$	$-(1+\nu)\xi_k(\phi)$	$-(1+\nu)\dfrac{d\xi_k}{d\phi}$	$-C_k(\phi_s,\frac{\pi}{2})\cos\phi$	$C_k(\phi_s,\frac{\pi}{2})\sin\phi$
	Wet: $\phi_s \leqslant \phi \leqslant \frac{\pi}{2}$	$\Phi_k(a,\phi)-(1+\nu)\xi_k(\phi)$		$-A_k(\phi_s,\phi)Q_1(\cos\phi)-C_k(\phi,\frac{\pi}{2})\cos\phi$	$-A_k(\phi_s,\phi)\dfrac{dQ_1}{d\phi}+C_k(\phi,\frac{\pi}{2})\sin\phi$
	Lower Hemisphere Wet: $\frac{\pi}{2} \leqslant \phi \leqslant \pi$	$\Phi_k(a,\phi)-(1+\nu)\xi_k(\phi)$	$-(1+\nu)\dfrac{d\xi_k}{d\phi}$	$C_k(\frac{\pi}{2},\phi)\cos\phi+A_k(\phi,\pi)Q_1(\cos\phi)$	$-C_k(\frac{\pi}{2},\phi)\sin\phi+A_k(\phi,\pi)\dfrac{dQ_1}{d\phi}$
$\phi_s \geqslant \frac{\pi}{2}$	Lower Hemisphere Dry: $\frac{\pi}{2} \leqslant \phi \leqslant \phi_s$	$-(1+\nu)\xi_k(\phi)$	$-(1+\nu)\dfrac{d\xi_k}{d\phi}$	$A_k(\phi_s,\pi)Q_1(\cos\phi)$	$A_k(\phi_s,\pi)\dfrac{dQ_1}{d\phi}$
	Wet: $\phi_s \leqslant \phi \leqslant \pi$	$\Phi_k(a,\phi)-(1+\nu)\xi_k(\phi)$		$C_k(\phi_s,\phi)\cos\phi+A_k(\phi,\pi)Q_1(\cos\phi)$	$-C_k(\phi_s,\phi)\sin\phi+A_k(\phi,\pi)\dfrac{dQ_1}{d\phi}$

APPENDIX

EVALUATION OF THE INTEGRALS (41) AND (42)

From equations (37) and (41), it follows by symbolic operations

$$A_k(\phi_1,\phi) = \left(\int_{\phi_1}^{\phi} P(\cos\psi) \cos\psi \sin\psi \, d\psi - \frac{h}{a} \right)^{2k-1} \tag{A-1}$$

where

$$\int P_j(\cos\psi) \cos\psi \sin\psi \, d\psi = \sin^2\psi \, f_j(\psi) + \sin\psi \, \cos\psi \, f_j'(\psi) + C \tag{A-2}$$

$$f_j(\psi) = - \frac{P_j(\cos\psi)}{(j-1)(j+2)} \qquad ; \qquad j = 0, 2, 3, ... \tag{A-3}$$

$$f_1(\psi) = -\frac{1}{3} \cos\psi \, \ln \sin\psi \qquad , \tag{A-4}$$

as can be proven by differentiation.

Now, combining equations (A-1) and (A-2) results in

$$A_k(\phi_1,\phi) = \left[\sin^2\psi \, f(\psi) + \sin\psi \, \cos\psi \, f'(\psi) \Bigg|_{\phi_1}^{\phi} - \frac{h}{a} \right]^{2k-1}$$

or

$$A_k(\phi_1,\phi) = \sin^2\psi \left(f - \frac{h}{a} \right)^{2k-1} + \sin\psi \, \cos\psi \left(f' - \frac{h}{a} \right)^{2k-1} \Bigg|_{\phi_1}^{\phi} \qquad .$$

Using the notations

$$\left[f(\psi) - \frac{h}{a} \right]^{2k-1} = F_k(\psi) \tag{A-5}$$

and

$$A_k(\psi) = \sin^2\psi \; F_k(\psi) + \sin\psi \; \cos\psi \; F_k{}'(\psi) \qquad , \tag{A-6}$$

one obtains, finally,

$$A_k(\phi_1,\phi) = A_k(\phi) - A_k(\phi_1) \tag{A-7}$$

The integration as indicated in equation (42) yields

$$C_k(\phi_1,\phi) = C_k(\phi) - C_k(\phi_1) \tag{A-8}$$

where

$$C_k(\phi) = \frac{1}{\cos\phi} \left[Q_1(\cos\phi) \; A_k(\phi) + F_k(\phi) \right] \qquad . \tag{A-9}$$

In particular, from equations (A-6) and (A-9), respectively, it can be concluded

$$A_k(\pi) = 0 \tag{A-10}$$

$$C_k\left(\frac{\pi}{2}\right) = -F_k{}'\left(\frac{\pi}{2}\right) \qquad . \tag{A-11}$$

In the latter case, the numerator and denominator of equation (A-9) converge independently to zero if ψ approaches $\pi/2$. The limit is obtained by the preceding differentiation of numerator and denominator.

REFERENCES

1. Glaser, R.F.: Axisymmetric Oscillation of a Partially Liquid Filled Cylindrical Shell Container Having Hemispherical Bulkheads. NASA TND-7318, June 1973.

2. Balabukh, L.I. and Molchanov, A.G.: Axisymmetric Oscillations of a Spherical Shell Partially Filled with a Fluid. Mekhanika Tvergodo Tela, Sept.-Oct. 1967, p. 56-61 (In Russian).

3. Mikhlin, S.G.: Variational Methods in Mathematical Physics. The MacMillan Co., New York, 1964.

4. Timoshenko, S. and Woinowsky-Krieger, S.: Theory of Plates and Shells. McGraw-Hill Book. Co., Inc., New York, 1961.

5. Courant, R. and Hilbert, D.: Methods of Mathematical Physics. First English Edition, Vol. I., Interscience Publishers, Inc., New York, 1953, pp. 351 ff.

6. Korn, G.A. and Korn, T.M.: Mathematical Handbook for Scientists and Engineers. McGraw-Hill Book Co., Inc., New York, 1961, pp. 159, 175, 302, 722, and 723.

7. Abramowitz, M. and Stegun, I.A., Editors: Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables. National Bureau of Standards Applied Mathematics Series 55, June 1964, pp. 798 and 784.